

A GENERALIZED FILTER REGULARIZATION RESULT FOR SOME NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

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ABSTRACT. Despite the strong focus of regularization on ill-posed problems, the general construction of such methods has not been fully explored. Moreover, many previous studies cannot be clearly adapted to handle more complex scenarios, albeit the greatly increasing concerns on the improvement of wider classes. In this note, we rigorously study a general theory for filter regularized operators in a Hilbert space for nonlinear evolution equations which have occurred naturally in different areas of science. The starting point lies in problems that are in principle ill-posed with respect to the initial/final data—these basically include the Cauchy problem for nonlinear elliptic equations and the backward-in-time nonlinear parabolic equations. We derive general filters that can be used to stabilize those problems. Essentially, we establish the corresponding well-posed problem whose solution converges to the solution of the ill-posed problem. The approximation can be confirmed by the error estimates in the Hilbert space. This work improves very much many papers in the same line of field.

1. INTRODUCTION AND PROBLEM SETTINGS

Let \mathcal{A} be a positive, self-adjoint operator in a Hilbert space \mathcal{H} and we denote by $\{E(\lambda), \lambda > 0\}$ the spectral resolution of the identify associated to \mathcal{A} . For $0 \leq t \leq T$, let us also denote by $\hat{Q}(t, \lambda)$ and $\hat{S}(t, \lambda)$ the Borel functions satisfying $C_1 e^{t\lambda} \leq \hat{Q}(t, \lambda), \hat{S}(t, \lambda) \leq C_2 e^{t\lambda}$ for some $C_1, C_2 > 0$ and for each λ . Let $\mathbb{Q}(t, \mathcal{A})$ and $\mathbb{S}(t, \mathcal{A})$ be operators satisfying:

- For any $v \in \mathcal{H}$ in the form of $v = \int_0^\infty dE(\lambda) v$ then

$$\mathbb{Q}(t, \mathcal{A}) = \int_0^\infty \hat{Q}(t, \lambda) dE(\lambda) v, \quad \mathbb{S}(t, \mathcal{A}) = \int_0^\infty \hat{S}(t, \lambda) dE(\lambda) v,$$

- $\mathbb{Q}(0, \mathcal{A})$ is the identity operator.

In this note, we consider the problem of determining the concentration $\mathbf{u} \in C([0, T]; \mathcal{H})$ from initial data \mathbf{u}_0 for the following integral equation

$$(1.1) \quad \mathbf{u}(t) = \mathbb{Q}(t, \mathcal{A}) \mathbf{u}_0 + \int_0^t \mathbb{S}(t - \tau, \mathcal{A}) f(\tau, \mathbf{u}(\tau)) d\tau, \quad t \in [0, T],$$

where the reaction rate f is uniformly Lipschitz in \mathcal{H} , i.e. $\|f(t, w_1) - f(t, w_2)\|_{\mathcal{H}} \leq L_f \|w_1 - w_2\|_{\mathcal{H}}$ for some constant $L_f > 0$ independent of $t \in [0, T]$ and every pair $(w_1, w_2) \in \mathcal{H} \times \mathcal{H}$. In addition, we suppose $f(t, 0) \equiv 0$ for all $t \in [0, T]$ for ease of presentation.

Such an interesting equation is well-known to be ill-posed in the sense of Hadamard. In other words, it does not necessarily admit a solution, and even if there exists uniquely a solution, it does not depend continuously on the data. On the other hand, the challenge in real-world applications is not only based on the appearance of the nonlinear production terms, but also includes the measurement on the data \mathbf{u}_0 . In fact, it can be assumed in this sense by the presence of an approximation \mathbf{u}_0^ε satisfying

$$(1.2) \quad \|\mathbf{u}_0^\varepsilon - \mathbf{u}_0\|_{\mathcal{H}} \leq \varepsilon,$$

in which the constant $\varepsilon > 0$ represents the upper bound of the noise level in measurement.

Due to the above-mentioned ill-posedness, one usually employs the so-called regularization methods to designate corresponding well-posed problems whose solutions can approximate the solutions of the ill-posed problems under certain assumptions. The equation (1.1) considered here arises from:

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- The Cauchy problem for semi-linear elliptic equations ([9, 8]) in the context of reconstructing the temperature of a body from interior measurements:

$$(1.3) \quad \frac{d^2 \mathbf{u}(t)}{dt^2} = \mathcal{A} \mathbf{u}(t) + f(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \frac{d\mathbf{u}(0)}{dt} = 0,$$

- The semi-linear backward-in-time parabolic equations ([7, 5]) in the framework of the backward heat conduction problem, calculating the initial heat distribution from the heat distribution at some point in finite time:

$$(1.4) \quad \frac{d\mathbf{u}(t)}{dt} = \mathcal{A} \mathbf{u}(t) + f(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0.$$

More precisely, it can be implicitly recognized to (1.1) that if we consider the mild solutions of such problems, it will address $\mathbb{Q}(t, \mathcal{A}) = \cosh(t\mathcal{A}^{\frac{1}{2}})$ and $\mathbb{S}(t, \mathcal{A}) = \mathcal{A}^{-\frac{1}{2}} \sinh(t\mathcal{A}^{\frac{1}{2}})$ for (1.3), whilst $\mathbb{Q}(t, \mathcal{A}) = \mathbb{S}(t, \mathcal{A}) = e^{t\mathcal{A}}$ is provided from (1.4). Clearly, these unbounded operators present the catastrophic growth on the solution and that, once again, makes the arguments for studying the regularization become widespread and well-researched.

Furthermore, it is worth noting in view of physical phenomena models that the problems (1.3) and (1.4) can be of applications including the sine-Gordon equation modeling the Josephson effects in superconductivity ([3]), the Lane-Emden-Fowler type system arising in molecular biology ([2]), and further the backward ultra-parabolic problem in population dynamics and multi-parameter Brownian motion ([10, 6, 4]).

In order to establish the regularized solution for (1.1), we follow the strategy of regularization that replaces the unbounded operators $\mathbb{Q}(t, \mathcal{A})$ and $\mathbb{S}(t, \mathcal{A})$, respectively, by bounded operators, denoted by $\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A})$ and $\mathbf{S}_\varepsilon^\beta(t, \mathcal{A})$, with respect to the noise level ε . Formally, it can be represented as

$$(1.5) \quad \bar{\mathbf{u}}_\varepsilon^\beta(t) = \mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) \mathbf{u}_0^\varepsilon + \int_0^t \mathbf{S}_\varepsilon^\beta(t - \tau, \mathcal{A}) f(\tau, \bar{\mathbf{u}}_\varepsilon^\beta(\tau)) d\tau, \quad t \in [0, T],$$

in which $\beta := \beta(\varepsilon) > 0$ is called the regularization parameter.

The main aim of this note is thus to construct a general property of these bounded operators. We are also concerned with the well-posedness of the integral equation (1.5) and interested very much in how fast the corresponding solution approximates the exact solution. With this premise, this paper is structured as follows: In Section 2, we provide Definition 1 for the filter regularized operator and then apply it to prove the well-posedness of (1.5) as well as its approximation in Theorem 2. Afterwards, the proof of the theorem is delivered in Section 3. In Section 4, we give some relative discussion to close this note.

2. FILTER REGULARIZED OPERATORS: DEFINITION AND APPLICATIONS

Definition 1. The operators $\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A})$ and $\mathbf{S}_\varepsilon^\beta(t, \mathcal{A})$ are called **filter regularized operators** if there exist positive constants \tilde{M}_1 and \tilde{M}_2 such that

- The norm on $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is bounded for all $t \in [0, T]$, i.e.

$$\|\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A})\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \leq \tilde{M}_1 \gamma(t, \beta), \quad \|\mathbf{S}_\varepsilon^\beta(t, \mathcal{A})\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \leq \tilde{M}_2 \gamma(t, \beta),$$

- There exists a functional space \tilde{W} such that $\mathcal{H} \subset \tilde{W}$ and the error estimate $\|\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) - \mathbb{Q}(t, \mathcal{A})\|_{\mathcal{L}(\tilde{W}, \mathcal{H})}$ is of the order $\gamma^{-1}(T - t, \beta)$. Here, the function $\gamma : [0, T] \times (0, \infty)$ satisfies that

(1) For any $\beta > 0$ then

$$\gamma(0, \beta) = 1, \quad \lim_{\beta \rightarrow 0^+} \gamma(t, \beta) = \infty \quad \text{for all } t \in [0, T];$$

(2) For $\tau_1, \tau_2 > 0$ then

$$\gamma(\tau_1 + \tau_2, \beta) = \gamma(\tau_1, \beta) \gamma(\tau_2, \beta);$$

(3) For $\tau_1 \geq \tau_2 > 0$ then

$$\gamma(\tau_1 - \tau_2, \beta) = \gamma(\tau_1, \beta) \gamma^{-1}(\tau_2, \beta).$$

Theorem 2. Let $\beta > 0$ satisfy the following conditions:

$$\begin{cases} \lim_{\varepsilon \rightarrow 0^+} \gamma^{-1}(T, \beta) = 0, \\ \lim_{\varepsilon \rightarrow 0^+} \gamma(T, \beta) \varepsilon = K \in [0, \infty). \end{cases}$$

Then the integral equation (1.5) admits a unique solution $\bar{\mathbf{u}}_\varepsilon^\beta \in C([0, T]; \mathcal{H})$. Assume that (1.1) has a unique solution $\mathbf{u}(t)$ such that

$$\|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(t, \mathbf{u}(t))\|_{\tilde{W}} dt < \infty,$$

the following error estimate holds:

$$\|\bar{\mathbf{u}}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} \leq \gamma^{-1}(T - t, \beta) \left(\tilde{M}_1 \gamma(T, \beta) \varepsilon \|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(t, \mathbf{u}(t))\|_{\tilde{W}} dt \right) \exp(\tilde{M}_2 L_f t),$$

for all $t \in [0, T]$.

3. PROOF OF THEOREM 2

3.1. Existence and uniqueness. For $v \in C([0, T]; \mathcal{H})$, we consider the following function

$$(3.1) \quad \Phi(v)(t) := \mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) \mathbf{u}_0^\varepsilon + \int_0^t \mathbf{S}_\varepsilon^\beta(t - \tau, \mathcal{A}) f(\tau, v(\tau)) d\tau.$$

At this stage, it is not capable of applying directly the Banach fixed point theorem for this function if the time interval is not small enough. Fortunately, one may show that there exists $m_0 \in \mathbb{N}$ such that Φ^{m_0} is a contraction mapping. In fact, we shall prove by induction that for every $w_1, w_2 \in C([0, T]; \mathcal{H})$ and $m \in \mathbb{N}$, the following estimate holds:

$$(3.2) \quad \|\Phi^m(w_1)(t) - \Phi^m(w_2)(t)\|_{\mathcal{H}} \leq \left(\tilde{M}_2 L_f \gamma(T, \beta) \right)^m \frac{t^m}{m!} \|w_1 - w_2\|_{C([0, T]; \mathcal{H})}.$$

Let $m = 1$, one easily checks (3.2) holds from (3.1). Suppose that (3.2) holds for $m = M$, we shall prove that (3.2) is still true for $m = M + 1$. Indeed, recalling the definition of the filter regularized operator $\mathbf{S}_\varepsilon^\beta$ and using the global Lipschitz assumption acting on f , we see that

$$\begin{aligned} \|\Phi^{M+1}(w_1)(t) - \Phi^{M+1}(w_2)(t)\|_{\mathcal{H}} &\leq \int_0^t \|\mathbf{S}_\varepsilon^\beta(t - \tau, \mathcal{A})\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \|f(\tau, \Phi^M(w_1)(\tau)) - f(\tau, \Phi^M(w_2)(\tau))\|_{\mathcal{H}} d\tau \\ &\leq \tilde{M}_2 L_f \int_0^t \gamma(t - \tau, \beta) \|\Phi^M(w_1)(\tau) - \Phi^M(w_2)(\tau)\|_{\mathcal{H}} d\tau \\ &\leq \tilde{M}_2 L_f \gamma(T, \beta) \left(\tilde{M}_2 L_f \gamma(T, \beta) \right)^M \int_0^t \frac{\tau^M}{M!} \|w_1 - w_2\|_{C([0, T]; \mathcal{H})} d\tau \\ &\leq \left(\tilde{M}_2 L_f \gamma(T, \beta) \right)^{M+1} \frac{t^{M+1}}{(M+1)!} \|w_1 - w_2\|_{C([0, T]; \mathcal{H})}. \end{aligned}$$

Thus, (3.2) holds true for all $m \in \mathbb{N}$, and that proves Φ^{m_0} is a contraction mapping for some $m_0 \in \mathbb{N}$ by the limitation of the right-hand side of (3.2) as $m \rightarrow \infty$. Hence, Φ^{m_0} has a unique solution $\bar{\mathbf{u}}_\varepsilon^\beta \in C([0, T]; \mathcal{H})$ for each $\varepsilon > 0$. This completes the proof of the existence and uniqueness for (1.5).

3.2. Stability analysis. In order to obtain the upper bound in \mathcal{H} of the regularized solution $\bar{\mathbf{u}}_\varepsilon^\beta$, the direct way is to estimate each part on the right-hand side of the expression (1.5). By Definition 1 and the structural inequality

$$\|f(t, \bar{\mathbf{u}}_\varepsilon^\beta(t))\|_{\mathcal{H}} \leq L_f \|\bar{\mathbf{u}}_\varepsilon^\beta(t)\|_{\mathcal{H}} + \|f(t, 0)\|_{\mathcal{H}} \leq L_f \|\bar{\mathbf{u}}_\varepsilon^\beta(t)\|_{\mathcal{H}},$$

we arrive at

$$\begin{aligned} \|\bar{\mathbf{u}}_\varepsilon^\beta(t)\|_{\mathcal{H}} &\leq \|\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) \mathbf{u}_0^\varepsilon\|_{\mathcal{H}} + \int_0^t \|\mathbf{S}_\varepsilon^\beta(t-\tau, \mathcal{A}) f(\tau, \bar{\mathbf{u}}_\varepsilon^\beta(\tau))\|_{\mathcal{H}} d\tau \\ &\leq \tilde{M}_1 \gamma(t, \beta) \|\mathbf{u}_0^\varepsilon\|_{\mathcal{H}} + \tilde{M}_2 \int_0^t \gamma(t-\tau, \beta) L_f \|\bar{\mathbf{u}}_\varepsilon^\beta(\tau)\|_{\mathcal{H}} d\tau. \end{aligned}$$

Multiplying both sides of the above estimate by $\gamma^{-1}(t, \beta)$ and notice that $\gamma(t-\tau, \beta) = \gamma(t, \beta) \gamma^{-1}(\tau, \beta)$, we see that

$$\gamma^{-1}(t, \beta) \|\bar{\mathbf{u}}_\varepsilon^\beta(t)\|_{\mathcal{H}} \leq \tilde{M}_1 \|\mathbf{u}_0^\varepsilon\|_{\mathcal{H}} + \tilde{M}_2 L_f \int_0^t \gamma^{-1}(\tau, \beta) \|\bar{\mathbf{u}}_\varepsilon^\beta(\tau)\|_{\mathcal{H}} d\tau.$$

Using the Grönwall inequality, we gain

$$\gamma^{-1}(t, \beta) \|\bar{\mathbf{u}}_\varepsilon^\beta(t)\|_{\mathcal{H}} \leq \exp(\tilde{M}_2 L_f t) \tilde{M}_1 \|\mathbf{u}_0^\varepsilon\|_{\mathcal{H}},$$

which tells the dependence of the solution on the initial data for each noise level.

3.3. Convergence rate. In this part, we need the help of the regularized solution as the exact data \mathbf{u}_0 is considered. Such a solution, denoted by $\mathbf{U}_\varepsilon^\beta$, can be formulated similarly as (1.5):

$$(3.3) \quad \mathbf{U}_\varepsilon^\beta(t) = \mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) \mathbf{u}_0 + \int_0^t \mathbf{S}_\varepsilon^\beta(t-\tau, \mathcal{A}) f(\tau, \mathbf{U}_\varepsilon^\beta(\tau)) d\tau, \quad t \in [0, T].$$

Thanks to the proof of the stability analysis, we proceed the same to obtain the difference estimate between $\mathbf{U}_\varepsilon^\beta(t)$ and $\bar{\mathbf{u}}_\varepsilon^\beta(t)$, as follows:

$$(3.4) \quad \|\bar{\mathbf{u}}_\varepsilon^\beta(t) - \mathbf{U}_\varepsilon^\beta(t)\|_{\mathcal{H}} \leq \exp(\tilde{M}_2 L_f t) \tilde{M}_1 \gamma(t, \beta) \|\mathbf{u}_0^\varepsilon - \mathbf{u}_0\|_{\mathcal{H}}.$$

From now on, recall that the difference $\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) - \mathbf{Q}(t, \mathcal{A})$ is of the order $\gamma^{-1}(T-t, \beta)$ uniformly in time. Therefore, one may get the difference estimate between $\mathbf{u}(t)$ and $\mathbf{U}_\varepsilon^\beta(t)$ by just subtracting them term by term. It follows from (1.1) and (3.3) that

$$\begin{aligned} \|\mathbf{U}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} &\leq \|(\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) - \mathbf{Q}(t, \mathcal{A})) \mathbf{u}_0\|_{\mathcal{H}} + \int_0^t \|\mathbf{S}_\varepsilon^\beta(t-\tau, \mathcal{A}) [f(\tau, \mathbf{U}_\varepsilon^\beta(\tau)) - f(\tau, \mathbf{u}(\tau))]\|_{\mathcal{H}} d\tau \\ &\quad + \int_0^t \|(\mathbf{S}_\varepsilon^\beta(t-s, \mathcal{A}) - \mathbf{S}(t-s, \mathcal{A})) f(s, \mathbf{u}(s))\|_{\mathcal{H}} ds \\ &\leq \gamma^{-1}(T-t, \beta) \|\mathbf{u}_0\|_{\tilde{W}} + \tilde{M}_2 L_f \int_0^t \gamma(t-\tau, \beta) \|\mathbf{U}_\varepsilon^\beta(\tau) - \mathbf{u}(\tau)\|_{\mathcal{H}} d\tau \\ &\quad + \gamma^{-1}(T-t, \beta) \int_0^T \|f(s, \mathbf{u}(s))\|_{\tilde{W}} ds. \end{aligned}$$

Multiplying both sides of the above estimate by $\gamma(T-t, \beta)$ in combination with the structural property $\gamma(T-t, \beta) \gamma(t-\tau, \beta) = \gamma(T-\tau, \beta)$, the resulting estimate can be thus written by

$$\begin{aligned} \gamma(T-t, \beta) \|\mathbf{U}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} &\leq \|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(s, \mathbf{u}(s))\|_{\tilde{W}} ds \\ &\quad + \tilde{M}_2 L_f \int_0^t \gamma(T-\tau, \beta) \|\mathbf{U}_\varepsilon^\beta(\tau) - \mathbf{u}(\tau)\|_{\mathcal{H}} d\tau. \end{aligned}$$

Once again, we apply the Grönwall inequality to gain that

$$(3.5) \quad \|\mathbf{U}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} \leq \gamma^{-1}(T-t, \beta) \left(\|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(s, \mathbf{u}(s))\|_{\tilde{W}} ds \right) \exp(\tilde{M}_2 L_f t).$$

At this moment, combining (3.4) and (3.5) in accordance with the assumption (1.2), we conclude that

$$\|\bar{\mathbf{u}}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} \leq \exp\left(\tilde{M}_2 L_f t\right) \tilde{M}_1 \gamma(t, \beta) \varepsilon + \gamma^{-1}(T - t, \beta) \left(\|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(s, \mathbf{u}(s))\|_{\tilde{W}} ds\right) \exp\left(\tilde{M}_2 L_f t\right),$$

which leads to the desired error estimate.

Hence, this completes the proof of the theorem.

4. DISCUSSION

As is known, the nonlinear reaction rate f is locally Lipschitz in real-world applications, i.e. for each $\mathcal{E} > 0$, there exists $L(\mathcal{E}) > 0$ such that

$$(4.1) \quad \|f(t, w_1) - f(t, w_2)\|_{\mathcal{H}} \leq L(\mathcal{E}) \|w_1 - w_2\|_{\mathcal{H}} \text{ as } \max\{\|w_1\|_{\mathcal{H}}, \|w_2\|_{\mathcal{H}}\} \leq \mathcal{E}.$$

Interestingly, our construction in this work can be also applicable to this case. In fact, since the above quantity $L(\mathcal{E})$ increases in $[0, \infty)$, we then choose a positive sequence $\{\mathcal{B}_\varepsilon\}_{\varepsilon > 0}$ satisfying $\lim_{\varepsilon \rightarrow 0^+} \mathcal{B}_\varepsilon = \infty$ and define the function $f_{\mathcal{B}_\varepsilon}$ as follows:

$$f_{\mathcal{B}_\varepsilon}(t, w) := f\left(t, \min\left\{\frac{\mathcal{B}_\varepsilon}{\|w\|_{\mathcal{H}}}, 1\right\} w\right) \quad \text{for } t \in [0, T], w \in \mathcal{H}.$$

Consequently, one can prove for ε small enough that $\|\mathbf{u}\|_{C([0, T]; \mathcal{H})} \leq \mathcal{B}_\varepsilon$, $f_{\mathcal{B}_\varepsilon}(t, \mathbf{u}(t)) = f(t, \mathbf{u}(t))$ for all $t \in [0, T]$ and the global Lipschitz property of $f_{\mathcal{B}_\varepsilon}$, i.e.

$$\|f(t, w_1) - f(t, w_2)\|_{\mathcal{H}} \leq 2L(\mathcal{B}_\varepsilon) \|w_1 - w_2\|_{\mathcal{H}}.$$

At this moment, we may repeat the proof of Theorem 2 to obtain the extended result on the locally Lipschitz case. We thus provide below the following theorem while skipping the proof.

Theorem 3. *Suppose that f is locally Lipschitz satisfying (4.1) and let β be as in Theorem 2. For each $\varepsilon > 0$, choose \mathcal{B}_ε such that*

$$\lim_{\varepsilon \rightarrow 0^+} \gamma^{-1}(T - t, \beta) \exp\left(2\tilde{M}_2 L(\mathcal{B}_\varepsilon) t\right) = 0 \quad \text{for all } t \in [0, T],$$

then the regularized solution that obeys the following integral equation

$$\tilde{\mathbf{u}}_\varepsilon^\beta(t) = \mathbf{Q}_\varepsilon^\beta(t, \mathcal{A}) \mathbf{u}_0^\varepsilon + \int_0^t \mathbf{S}_\varepsilon^\beta(t - \tau, \mathcal{A}) f_{\mathcal{B}_\varepsilon}(\tau, \tilde{\mathbf{u}}_\varepsilon^\beta(\tau)) d\tau,$$

exists uniquely in $C([0, T]; \mathcal{H})$. Furthermore, assume that (1.1) has a unique solution $\mathbf{u}(t)$ as in Theorem 2, then the following error estimate holds:

$$\|\tilde{\mathbf{u}}_\varepsilon^\beta(t) - \mathbf{u}(t)\|_{\mathcal{H}} \leq \gamma^{-1}(T - t, \beta) \left(\tilde{M}_1 \gamma(T, \beta) \varepsilon \|\mathbf{u}_0\|_{\tilde{W}} + \int_0^T \|f(t, \mathbf{u}(t))\|_{\tilde{W}} dt\right) \exp\left(2\tilde{M}_2 L(\mathcal{B}_\varepsilon) t\right).$$

It is worth noting that these extensions not only serve fast growing rates (e.g. the Van der Pole type nonlinearity $f(u) = u^3 - u$ in single-species case), but also include Arrhenius-like laws (i.e. exponential rates of the type $f(u) = \exp(|u|)$). On top of that, this work can be applied very similarly to pretty much wider classes. In particular, the same approximation can be done with the strongly damped semi-linear wave problems ([1]):

$$(4.2) \quad \frac{d^2 \mathbf{u}(t)}{dt^2} + \mathcal{A} \left(\mathbf{u}(t) + \frac{d\mathbf{u}(t)}{dt} \right) = f(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \frac{d\mathbf{u}(0)}{dt} = 0,$$

In this regards, we compute for (4.2) that

$$\mathbb{Q}(t, \mathcal{A}) = \frac{\chi_+(\mathcal{A}) \exp(\chi_-(\mathcal{A})t) - \chi_-(\mathcal{A}) \exp(\chi_+(\mathcal{A})t)}{\chi_+(\mathcal{A}) - \chi_-(\mathcal{A})}, \quad \mathbb{S}(t, \mathcal{A}) = \frac{\exp(\chi_-(\mathcal{A})t) - \exp(\chi_+(\mathcal{A})t)}{\chi_-(\mathcal{A}) - \chi_+(\mathcal{A})},$$

with $\chi_+(\mathcal{A}) = 0.5 \left(-\mathcal{A} + (\mathcal{A}^2 - 4\mathcal{A})^{\frac{1}{2}} \right)$ and $\chi_-(\mathcal{A}) = 0.5 \left(-\mathcal{A} - (\mathcal{A}^2 - 4\mathcal{A})^{\frac{1}{2}} \right)$.

We remark that the presence of the non-homogeneous initial velocity in (1.3) and (4.2) will not also change the result of this research. Let $\mathbf{u}_1 \in \mathcal{H}$ be the time derivative of the concentration at $t = 0$, this circumstance leads us to the following mild solution:

$$\mathbf{u}(t) = \mathbb{Q}(t, \mathcal{A}) \mathbf{u}_0 + \mathbb{S}(t, \mathcal{A}) \mathbf{u}_1 + \int_0^t \mathbb{S}(t - \tau, \mathcal{A}) f(\tau, \mathbf{u}(\tau)) d\tau, \quad t \in [0, T],$$

which is analogous to (1.1).

Finally, several examples for the regularized operators $\mathbf{Q}_\varepsilon^\beta(t, \mathcal{A})$ and $\mathbf{S}_\varepsilon^\beta(t, \mathcal{A})$ can be found very easily, e.g. in [7, 8] whereas $\gamma(t, \beta) = \beta^{-\frac{t}{T}}$ is pointed out therein.

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